Exercise 1

Solve the differential equation or initial-value problem using the method of undetermined coefficients.

$$y'' + 2y' - 8y = 1 - 2x^2$$

Solution

Since the ODE is linear, the general solution can be written as the sum of a complementary solution and a particular solution.

$$y = y_c + y_p$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + 2y_c' - 8y_c = 0 \tag{1}$$

This is a linear homogeneous ODE, so its solutions are of the form $y_c = e^{rx}$.

$$y_c = e^{rx} \quad \rightarrow \quad y_c' = re^{rx} \quad \rightarrow \quad y_c'' = r^2 e^{rx}$$

Plug these formulas into equation (1).

$$r^2 e^{rx} + 2(re^{rx}) - 8(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 + 2r - 8 = 0$$

Solve for r.

$$(r + 4)(r - 2) = 0$$

$$r = \{-4, 2\}$$

Two solutions to the ODE are e^{-4x} and e^{2x}; by the principle of superposition, then,

$$y_c(x) = C_1 e^{-4x} + C_2 e^{2x}.$$

The particular solution satisfies the original ODE.

$$y_p'' + 2y_p' - 8y_p = 1 - 2x^2 \tag{2}$$

Since the inhomogeneous term is a polynomial of degree 2, the particular solution is $y_p = Ax^2 + Bx + C$.

$$y_p = Ax^2 + Bx + C \quad \rightarrow \quad y_p' = 2Ax + B \quad \rightarrow \quad y_p'' = 2A$$

Substitute these formulas into equation (2).

$$2A + 2(2Ax + B) - 8(Ax^2 + Bx + C) = 1 - 2x^2$$

$$(2A + 2B - 8C) + (4A - 8B)x - 8Ax^2 = 1 - 2x^2$$

Match the coefficients on both sides to get a system of equations for A, B, and C.

$$\left.\begin{aligned} 2A + 2B - 8C &= 1 \\ 4A - 8B &= 0 \\ -8A &= -2 \end{aligned}\right\}$$

Solving this system yields

$$A = \frac{1}{4} \quad \text{and} \quad B = \frac{1}{8} \quad \text{and} \quad C = -\frac{1}{32},$$

which means the particular solution is

$$y_p = \frac{1}{4}x^2 + \frac{1}{8}x - \frac{1}{32}.$$

Therefore, the general solution to the ODE is

$$\begin{aligned} y(x) &= y_c + y_p \\ &= C_1 e^{-4x} + C_2 e^{2x} + \frac{1}{4}x^2 + \frac{1}{8}x - \frac{1}{32}, \end{aligned}$$

where C_1 and C_2 are arbitrary constants.